

Partition Parameters for Girth Maximum (m, r) BTUs

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Abstract—This paper describes the calculation of the optimal partition parameters such that the girth maximum (m, r) Balanced Tanner Unit lies in family of BTUs specified by them using a series of proved results and thus creates a framework for specifying a search problem for finding the girth maximum (m, r) BTU. Several open questions for girth maximum (m, r) BTU have been raised.

Keywords *Permutation Groups, Cycle Index, Balanced Tanner Unit, Girth Maximum (m, r) BTU*

1 Introduction

We have introduced a family of bi-partite graphs called Balanced Tanner Units (BTUs) in [1] and have introduced a family of graphs $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$. The goal of this paper is to derive the forms for optimal partitions $\beta_1, \beta_2, \dots, \beta_{r-1}$ such that the girth maximum (m, r) BTU lies in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ using a series of mathematical results that build upon the fundamentals of BTUs introduced in [1].

We review a few definitions from [1].

1.1 Set $P_2(m)$

$P_2(m)$ refers to the set of partitions of $m \in \mathbb{N}$ that consist of numbers that are greater than or equal to 2.

1.2 Partition Component

If $\beta \in P_2(m)$ refers to $\sum_{j=1}^y q_j = m$ then each $\{q_j\}$ for $1 \leq j \leq y$ is referred to as a partition component of β .

1.3 Definition of $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$

$\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ refers to the family of all labeled (m, r) BTUs with compatible permutations $p_1, p_2, \dots, p_r \in S_m; p_i \notin C(p_1, p_2, \dots, p_{i-1})$ for $1 < i \leq r$ that occur in the same order on a complete m symmetric permutation tree, $x_{1,1} < x_{2,1} < \dots < x_{r,1}$ where $p_j = (x_{j,1}x_{j,2} \dots x_{j,m}); 1 \leq j \leq r$, such that β_{i-1} is the partition between permutations p_{i-1} and p_i for all integer values of i given by $1 < i \leq r$.

1.4 Symmetric Permutation Tree and its properties

A m Symmetric permutation tree $S_{PT}\{m\}$ is defined as a labeled tree with the following properties:

1. $S_{PT}\{m\}$ has a single root node labeled 0.
2. $S_{PT}\{m\}$ has m nodes at depth 1 from the root node.
3. $S_{PT}\{m\}$ has nodes at depths ranging from 1 to m , with each node having a labels chosen from $\{1, 2, \dots, m\}$. The root node 0 has m successor nodes. Each node at depth 1 has $m-1$ successor nodes at depth 2. Each node at depth i has $m-i+1$ successor nodes at depth $i+1$. Each node at depth $m-1$ has 1 successor node at depth m .
4. No successor node in $S_{PT}\{m\}$ has the same node label as any of its ancestor nodes.
5. No two successor nodes that share a common parent node have the same label.
6. The sequence of nodes in the path traversal from the node at depth 1 to the leaf node at depth m in $S_{PT}\{m\}$ represents the permutation represented by the leaf node.
7. $S_{PT}\{m\}$ has $m!$ leaf nodes each of which represent an element of the symmetric group of degree m denoted by S_m .

1.5 Compatible Permutations

1. Two permutations on a set of s elements represented by $(x_1x_2 \dots x_s); x_p \neq x_q \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N}$ where $1 \leq x_i \leq s; i \in \mathbb{N}; 1 \leq i \leq s$ and $(y_1y_2 \dots y_s); y_p \neq y_q \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N}$ where $1 \leq y_i \leq m; i \in \mathbb{N}; 1 \leq i \leq s$ are compatible if and only if $x_i \neq y_i \forall i \in \mathbb{N}; 1 \leq i \leq s$.
2. A set of r permutations on a set of s elements represented by $(x_{i,1}x_{i,2} \dots x_{i,s}); x_{i,p} \neq x_{i,q} \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N}$ where $1 \leq x_{i,\alpha} \leq s \forall 1 \leq i \leq r; 1 \leq \alpha \leq s; i, \alpha \in \mathbb{N}$ are compatible if and only if $x_{i,\alpha} \neq x_{j,\alpha} \forall i \neq j; 1 \leq \alpha \leq s; 1 \leq i \leq r; 1 \leq j \leq r; i, j, \alpha \in \mathbb{N}$.

1.6 Notation

$p_i \notin C(I_m, p_2, \dots, p_{i-1}) : p_i$ is compatible with permutations I_m, p_2, \dots, p_{i-1} .

2 General Approach at Constructing (m, r) BTU with the maximum girth

2.1 Conjecture

Any non-isomorphic (m, r) BTU can be constructed by choosing compatible permutations $p_1, p_2, \dots, p_r \in S_m; p_{j+1} \notin C(p_1, p_2, \dots, p_j)$ for $1 \leq j \leq r-1$ that occur in the same order on a m complete symmetric permutation tree, $x_{1,1} < x_{2,1} < \dots < x_{r,1}$ i.e., $x_{j,1} < x_{j+1,1}$ for $1 \leq j \leq r-1$, where $p_i = (x_{i,1}x_{i,2} \dots x_{i,m})$ for $1 \leq i \leq r$.

Proof Without loss of generality, any labeled (m, r) BTU can be represented by a choice of permutations $p_1, p_2, \dots, p_r \in S_m; p_{j+1} \notin C(p_1, p_2, \dots, p_j)$ for $1 \leq j \leq r-1$ that occur in the same order on a m complete symmetric permutation tree, $x_{1,1} < x_{2,1} < \dots < x_{r,1}$ and hence any non-isomorphic (m, r) BTU can be constructed by choosing compatible permutations $p_1, p_2, \dots, p_r \in S_m; p_{j+1} \notin C(p_1, p_2, \dots, p_j)$ for $1 \leq j \leq r-1$ that occur in the same order on a m complete symmetric permutation tree, $x_{1,1} < x_{2,1} < \dots < x_{r,1}$.

2.2 Conjecture

Any non-isomorphic (m, r) BTU can be constructed by choosing compatible permutations $p_1, p_2, \dots, p_r \in S_m; p_{j+1} \notin C(p_1, p_2, \dots, p_j)$ for $1 \leq j \leq r-1$ such that $x_{j,1} = j$ for $1 \leq j \leq r$, where $p_i = (x_{i,1}x_{i,2} \dots x_{i,m})$ for $1 \leq i \leq r$.

Proof Without loss of generality, any non-isomorphic (m, r) BTU is isomorphic to a BTU represented by a set of permutations $\{p_1, p_2, \dots, p_r\}$ where $p_1, p_2, \dots, p_r \in S_m; p_{j+1} \notin C(p_1, p_2, \dots, p_j)$ for $1 \leq j \leq r-1$ such that $x_{j,1} = j$ for $1 \leq j \leq r$, since it could be brought to the form by row exchanges or permutations on depth for the permutation representation of the labeled (m, r) BTU.

2.3 Theorem

Any (m, r) BTU is isomorphic to an element of $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ for some choice of $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$.

Proof In general any (m, r) BTU can be diagonalized by suitable row and column exchanges, and without loss of generality, $p_1 = I_m$. We map the positions of 1s in the first column to $x_{i,1}; 2 \leq i \leq r$ where $p_i = (x_{i,1}x_{i,2} \dots x_{i,m})$. Thus, map the (m, r) BTU to

compatible permutations p_1, p_2, \dots, p_r that occur in the same order on a complete m symmetric permutation tree. Let $\beta_i \in P_2(m)$ be the partition represented between p_i and p_{i+1} for $1 \leq i \leq r-1$. Hence, any (m, r) BTU is isomorphic to an element of $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ for some choice of $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$.

2.4 Implicit Enumeration Conjecture

All non-isomorphic (m, r) BTUs can be enumerated by

1. An exhaustive enumeration of $\{\beta_1, \beta_2, \dots, \beta_{r-1}\}$ where $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$.
2. For each of the above enumeration of $\{\beta_1, \beta_2, \dots, \beta_{r-1}\}$ we construct all non-isomorphic (m, r) BTU with β_i being the partition between p_{i-1} and p_i for all integer values of i given by $2 \leq i \leq r-1$ represented by the set $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$.

2.5 Definition of Micro-partition

Given partitions $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ with β_i of the form $\sum_{j=1}^{y_i} p_{i,j} = m; i \in \mathbb{N}, 1 \leq i \leq r-1$, a micro-partition of β_{i+1} with respect to β_i is defined as $x_{i,j,z} \in \mathbb{N} \cup \{0\}; 1 \leq i < r-1; 1 \leq j \leq y_i; 1 \leq z \leq y_{i+1}$ such that $\sum_{z=1}^{y_{i+1}} x_{i,j,z} = p_{i,j}; 1 \leq j \leq y_i; 1 \leq i < r-1$ and $\sum_{j=1}^{y_i} x_{i,j,z} = p_{i+1,z}; 1 \leq z \leq y_{i+1}; 1 \leq i < r-1$.

2.6 Micro-partition to label mapping

For each micro-partition of β_{i+1} with respect to β_i , $x_{i,j,z} \in \mathbb{N} \cup \{0\}; 1 \leq i < r-1; 1 \leq j \leq y_i; 1 \leq z \leq y_{i+1}$ we define **Micro-partition to label mapping** $y(x_{i,j,z})$ as an ordered set of $x_{i,j,z}$ distinct labels.

1. For β_1 , the labels from $\{1, 2, \dots, m\}$ for each partition component gets fixed by $\Psi(\beta_1)$.
2. For $\beta_2, \dots, \beta_{r-1}$ each distinct choice of a micro-partition to label mapping leads to numerous labeled graphs, which in turn would correspond to elements in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$.

2.7 Algorithm to enumerate all non-isomorphic (m, r) BTUs for $r \geq 3$

1. Enumerate all distinct ordered combinations of $\{\beta_1, \beta_2, \dots, \beta_{r-1}\}$ where $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$.
2. For each instance of $\{\beta_1, \beta_2, \dots, \beta_{r-1}\}$, we enumerate all unique combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$.
3. For each instance of a combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, we enumerate all sets of $r-1$ Unordered labeled partitions.

4. For each instance of a set of $r-1$ Unordered labeled partitions, we enumerate all possible sets of ordered $r-1$ Ordered labeled partitions using Reduced Cycle Enumeration corresponding to each Unordered labeled partition enumerated in the previous step.
5. We construct p_2, p_3, \dots, p_r corresponding to the set of $r-1$ Ordered labeled partitions enumerated in the previous step. ($p_1 = I_m$).

We shall develop the rationale for the this algorithm in subsequent sections.

2.8 Algorithm α

A labeled (m, r) BTU can be characterized by a set of compatible permutations $p_1, p_2, \dots, p_r \in S_m$ such that $p_i \notin C(p_1, \dots, p_{i-1}); 1 < i \leq r$.

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 $p_1 = I_m$  ;
for(each possible  $p_2; p_2 \notin C(p_1)$  ) {
for(each possible  $p_3; p_3 \notin C(p_1, p_2)$  ) {
...
for(each possible  $p_r; p_r \notin C(p_1, p_2, \dots, p_{r-1})$  ) {
for(each chosen  $p_1, p_2, \dots, p_r \in S_m$  such that  $p_i \notin C(p_1, \dots, p_{i-1}); 1 < i \leq r$ )
We evaluate the girth of the chosen  $(m, r)$  BTU;
}
...
}
}

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Choose $p_1, p_2, \dots, p_r \in S_m$ with the best girth from the above explorations.

2.9 Theorem

Algorithm α chooses a (m, r) BTU with the best girth. **Proof** Even though the algorithm does not enumerate all labeled (m, r) BTUs since we choose $p_1 = I_m$, it clearly covers the space of all non-isomorphic (m, r) BTUs with a huge amount of duplications since any labeled (m, r) BTU is an element of $\Psi(\beta_1, \beta_2, \dots, \beta_{r-1})$ for some $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$. The algorithm clearly chooses a (m, r) BTU with the best girth from elements of all possible sets $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$.

2.10 Algorithm α_1

Enumerate all distinct possible $\{\beta_1, \beta_2, \dots, \beta_{r-1}\}$ such that $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$;

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for(each enumeration of  $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ ) {
 $p_1 = I_m$  ;
for(each possible  $p_2; \beta_1(p_1, p_2); p_2 \notin C(p_1)$  ) {
for(each possible  $p_3; \beta_2(p_2, p_3); p_3 \notin C(p_1, p_2)$  ) {
...
for(each possible  $p_r; \beta_{r-1}(p_r, p_{r-1}); p_r \notin C(p_1, p_2, \dots, p_{r-1})$ 

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} {
We evaluate the girth of the chosen  $(m, r)$  BTU represented by the chosen permutations  $p_1, p_2, \dots, p_r \in S_m$  such that  $p_i \notin C(p_1, \dots, p_{i-1}); 1 < i \leq r$ ;
}
...
}
}

```

Choose $p_1, p_2, \dots, p_r \in S_m$ with the best girth from the above explorations.

2.11 Theorem

Algorithm α_1 chooses a (m, r) BTU with the best girth in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ given $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$.

Proof The Algorithm α_1 clearly covers the space of all non-isomorphic (m, r) BTUs in the family $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$. Hence, the chosen $p_1, p_2, \dots, p_r \in S_m$ by Algorithm α_1 represents a (m, r) BTU with the best girth in the family $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$.

2.12 Ordered labeled Partition

An ordered labeled Partition of m consists of distinct ordered y subsets B_1, B_2, \dots, B_y of the set $A = \{1, 2, \dots, m\}$ such that

1. The ordered subsets satisfy the condition $B_i \cap B_j = \Phi$ the null set $\forall i \neq j, 1 \leq i \leq y; 1 \leq j \leq y$
2. Each **ordered** set $B_i \subset A$ with number of distinct elements $p_i; 1 \leq i \leq y$.

2.13 Unordered labeled Partition

Given $\beta \in P_2(m)$ which refers to $\sum_{i=1}^y p_i = m$, an unordered labeled partition of m is a collection of distinct y subsets B_1, B_2, \dots, B_y of the set $A = \{1, 2, \dots, m\}$ such that

1. $B_i \cap B_j = \Phi$ the null set $\forall i \neq j, 1 \leq i \leq y; 1 \leq j \leq y$
2. Each set $B_i \subset A$ with number of distinct elements $p_i; 1 \leq i \leq y$.

2.14 Unordered labeled Partition Mapping Enumeration Problem

Given $\beta \in P_2(m)$ which refers to $\sum_{i=1}^y p_i = m$, the unordered labeled partition mapping Enumeration problem refers to the Enumeration of all distinct y subsets B_1, B_2, \dots, B_y of the set $A = \{1, 2, \dots, m\}$ such that

1. $B_i \cap B_j = \Phi$ the null set $\forall i \neq j, 1 \leq i \leq y; 1 \leq j \leq y$

- Each set $B_i \subset A$ with number of distinct elements $p_i; 1 \leq i \leq y$.

2.15 Ordered labeled Partition Mapping Enumeration Problem

Given $\beta \in P_2(m)$ which refers to $\sum_{i=1}^y p_i = m$, the **ordered** labeled partition mapping Enumeration problem refers to the Enumeration of all distinct **ordered** y subsets B_1, B_2, \dots, B_y of the set $A = \{1, 2, \dots, m\}$ such that

- $B_i \cap B_j = \Phi$, the null set $\forall i \neq j, 1 \leq i \leq y; 1 \leq j \leq y$.
- Each ordered set $B_i \subset A$ with number of distinct elements $p_i; 1 \leq i \leq y$.

2.16 Partitions \rightarrow micro-partition \rightarrow Un-ordered labeled Partitions \rightarrow Ordered labeled Partitions \rightarrow Compatible Permutations \rightarrow labeled (m, r) BTUs

Given a set of partitions $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$, we can have many possible sets of micro-partitions. For each set of micro-partitions, we can have many possible unordered labeled partitions. For each set of unordered labeled partitions, we can have many possible ordered labeled partitions. For each set of $r-1$ ordered labeled partitions, we can construct a set of compatible permutations

$\{I_m, p_2, \dots, p_{r-1}\}$ which represents a labeled (m, r) BTU.

2.17 Multiple levels of enumeration

Depth Permutation & Label permutations that preserve p_1 and p_2 and create different instances of p_3 for the same

- Enumeration of all possible micro-partitions of β_{i+1} w.r.t. β_i .
- For each micro-partition, enumeration of all possible non-ordered labeled partitions corresponding to β_{i+1} .
- For each non-ordered labeled partition, enumerate all cycle orders.

2.18 Constrained labeled Partition Mapping Problem

The Permutation Enumeration formulae derived in [1] gives us the number of candidate permutations corresponding to a specified partition. We recall from [1] that

the number of distinct permutations $\{p_2; \beta_1(p_2, p_1) = (m), p_2 \notin C(p_1)\}$ is given by

$$f(\beta) = (m-1) * \sum_{j, \text{distinct } p_j} \frac{(m-1)!}{(p_{1,j}-1)! * \prod_{i=1, i \neq j}^{y_1} (p_{1,i})}.$$

For $\beta = (m)$; we obtain $(m-1)!$ distinct permutations on a m symmetric permutation tree, using the above formula.

2.19 Conjecture

Any labeled $(m, 2)$ BTU with the first permutation $p_1 = I_m$ can be represented by a set of ordered labeled partitions on m elements.

Proof Since any labeled $(m, 2)$ BTU is isomorphic to $\Psi(\beta)$ for some $\beta \in P_2(m)$, any labeled $(m, 2)$ BTU with the first permutation $p_1 = I_m$ and second permutation $p_2; p_2 \notin C(p_1); \beta(p_1, p_2); p_1 = I_m$ can be mapped to an ordered labeled partition $K(\beta, p_1)$ on m in the following manner. Each ordered subset in $K(\beta, p_1)$ consists of q_i elements of distinct labels $\{l_{i,1}, l_{i,2}, \dots, l_{i,y_i}\}$ such that labels $l_{i,1}$ in p_1 and $l_{i,2}$ in p_2 are located at the same depth, labels $l_{i,2}$ in p_1 and $l_{i,3}$ in p_2 are located at the same depth, ..., and finally, labels l_{i,y_i} in p_1 and $l_{i,1}$ in p_2 are located at the same depth. Thus,

$$K(\beta, p_1) \text{ has the form } (l_{1,1}, \dots, l_{1,y_1}), (l_{2,1}, \dots, l_{2,y_1}), \dots, (l_{y_1,1}, \dots, l_{y_1,y_1}).$$

2.20 Corollary

Given an ordered labeled partition on m represented by $K(\beta, p_1)$, any circular permutation on ordered subsets yields the same labeled $(m, 2)$ BTU.

2.21 Corollary

Given an ordered labeled partition on m represented by $K(\beta, p_1)$, with first permutation $p_1 = I_m$, p_2 gets precisely defined resulting in a unique labeled $(m, 2)$ BTU.

2.22 Conjecture

Any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ ordered labeled partitions on m elements.

Proof Let the given labeled (m, r) BTU consists of permutations $p_1 = I_m$ and $p_{i+1}; p_{i+1} \notin C(p_1, p_2, \dots, p_i); p_1 = I_m; 2 \leq i \leq r-1$. Starting with $p_1 = I_m$, the compatible permutation p_2 can be represented as an ordered labeled partition $K_1(\beta_1, p_1)$ on m in the following manner. Each ordered subset in $K_1(\beta_1, p_1)$ consists of q_1 elements of distinct labels $\{l_{1,1,1}, l_{1,1,2}, \dots, l_{1,1,y_1}\}$ such that labels $l_{1,1,1}$ in p_1 and $l_{1,1,2}$ in p_2 are located at the same depth, labels $l_{1,1,2}$ in p_1 and $l_{1,1,3}$ in p_2 are located at the same depth, ...,

and finally, labels $l_{1,1,y_1}$ in p_1 and $l_{1,1,1}$ in p_2 are located at the same depth.

The compatible permutation $p_{i+1}; p_{i+1} \notin C(p_1, p_2, \dots, p_i); p_1 = I_m; 2 \leq i \leq r-1$ can be represented as an ordered labeled partition $K_i(\beta_i, p_i)$ on m in the following manner. Each ordered subset in $K_i(\beta_i, p_i)$ consists of $q_j; 1 \leq j \leq y_i$ elements of distinct labels $\{l_{i,j,1}, l_{i,j,2}, \dots, l_{i,j,y_i}\}$ such that labels $l_{i,j,1}$ in p_i and $l_{i,j,2}$ in p_{i+1} are located at the same depth, labels $l_{i,j,2}$ in p_i and $l_{i,j,3}$ in p_{i+1} are located at the same depth, ..., and finally, labels l_{i,j,y_i} in p_i and $l_{i,j,1}$ in p_{i+1} are located at the same depth.

Thus, any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ ordered labeled partitions on m elements represented by $\{K_i(\beta_i, p_i); 1 \leq i \leq r-1\}$.

Each $K_i(\beta_i, p_i)$ has the form $(l_{i,1,1}, l_{i,1,2}, \dots, l_{i,1,y_i}), (l_{i,2,1}, l_{i,2,2}, \dots, l_{i,2,y_i}), \dots, (l_{i,y_i,1}, l_{i,y_i,2}, \dots, l_{i,y_i,y_i})$.

2.23 Conjecture

A set of $r-1$ ordered labeled permutations on m represented by $\{K_i(\beta_i, p_i); 1 \leq i \leq r-1\}$ corresponds to a labeled (m, r) BTU, if the resultant permutations $\{p_{i+1}; 1 \leq i \leq r-1\}$ are compatible with $\{p_1, p_2, \dots, p_i; 1 \leq i \leq r-1\}$.

Proof This follows from the fact that permutations $\{p_1, p_2, \dots, p_r\}$ correspond to a labeled (m, r) BTU if and only if $p_{i+1} \notin C(p_1, p_2, \dots, p_i)$ for $1 \leq i \leq r-1$.

2.24 Conjecture

If the first permutation p_1 of a labeled (m, r) BTU is specified, and given set of $r-1$ ordered labeled permutations on m represented by $\{K_i(\beta_i, p_i); 1 \leq i \leq r-1\}$ such that the resultant permutations $\{p_{i+1}; 1 \leq i \leq r-1\}$ are compatible with $\{p_1, p_2, \dots, p_i; 1 \leq i \leq r-1\}$, then the labeled (m, r) BTU is unique.

Proof This follows from the fact that permutations $\{p_1, p_2, \dots, p_r\}$ correspond to a labeled (m, r) BTU if and only if $p_{i+1} \notin C(p_1, p_2, \dots, p_i)$ for $1 \leq i \leq r-1$ and since the first permutation p_1 is specified, we obtain p_2 using $\{K_{(\beta_1, p_1)}\}$. Similarly, we obtain p_{i+1} from p_i and $\{K_{(\beta_i, p_i)}\}$ for $1 \leq i \leq r-1$. We do not separately need an explicit specification of $\beta_i; 1 \leq i \leq r-1$ since it is already implicitly specified in the definitions of $\{K_i(\beta_i, p_i); 1 \leq i \leq r-1\}$.

2.25 Definition Of Cycle Order

Given an unordered labeled partition $U = B_1 \cup B_2 \cup \dots \cup B_y$ on m such that $B_i \cap B_j = \Phi$ the empty set for $i \neq j, 1 \leq i \leq y; 1 \leq j \leq y$, and each unordered subset B_i contains distinct elements of the set $\{1, 2, \dots, m\}$, a

cycle order on a subset B_i is an ordered subset C_i with the same elements such that $O = C_1 \cup C_2 \cup \dots \cup C_y$ is an ordered labeled partition on m .

2.26 Corollary

A defined cycle order on each unordered subset of an unordered labeled partition on m , gives us an ordered labeled partition on m .

2.27 Conjecture

Any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ sets of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, mapping from micro-partitions to labels, and defined cycle orders for each partition component.

Proof Any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ ordered labeled partitions on m . Each of the $r-1$ ordered labeled partitions could be represented by $r-1$ unordered labeled partitions and specific cycle orders for each subset given separately.

Each of the $r-1$ unordered labeled partitions could be uniquely specified by a set of $r-1$ sets of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, mapping from micro-partitions to labels. Hence, it follows that any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ sets of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, mapping from micro-partitions to labels, and defined cycle orders for each partition component.

2.28 Corollary

Given a set of permutations p_1, p_2, \dots, p_i where $p_1 = I_m$ and $p_{i+1} \notin C(p_1, p_2, \dots, p_i)$, each permutation satisfying the condition $\{p_{i+1}; \beta_i(p_{i+1}, p_i), p_{i+1} \notin C(p_1, \dots, p_i)\}$ can be represented by micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, i unordered labeled partitions corresponding to micro-partitions, and finally i ordered labeled partitions.

2.29 Micro-partitions Enumeration Problem

Given $\beta_{i+1}, \beta_i \in P_2(m)$, the micro-partition enumeration problem refers to enumeration of all possible micro-partitions of β_{i+1} w.r.t. β_i .

2.30 Micro-partitions Mapping Enumeration Problem

Given micro-partitions of β_{i+1} w.r.t. β_i and p_1, p_2, \dots, p_i to enumerate all possible **unordered** la-

beled partitions of m with each subset of labels corresponding to each of the cycles of β_{i+1} .

2.31 Conjecture Micro-partitions to Permutation Counting $\{p_3; \beta_2(p_3, p_2), p_3 \notin C(p_2, p_1 = I_m)\}$

Given micro-partitions $x_{1,j,z}$ of β_2 w.r.t. β_1 , the number of ways to map the micro-partitions to unordered labeled partitions is given by the following expression

$$\prod_{j=1}^{y_1} \prod_{z=1}^{y_2} \binom{p_{1,j} - \sum_{k=1}^{z-1} x_{1,j,k}}{x_{1,j,z}}.$$

Proof This follows directly from number of ways to choose $x_{1,j,z}$ elements from $p_{1,j} - \sum_{k=1}^{z-1} x_{1,j,k}$ elements for $1 \leq z \leq y_2$ and $1 \leq j \leq y_1$ which yields

$$\prod_{j=1}^{y_1} \prod_{z=1}^{y_2} \binom{p_{1,j} - \sum_{k=1}^{z-1} x_{1,j,k}}{x_{1,j,z}}.$$

2.32 Cycle Order Enumeration Problem

Given a set of $r-1$ unordered labeled partitions of m , to enumerate all possible distinct $r-1$ ordered labeled partitions, each of which refer a distinct choice of compatible permutation. $p_{i+1} \notin C(p_1, p_2, \dots, p_i)$ w.r.t. p_1, p_2, \dots, p_i for $1 \leq i \leq r-1$. Each of enumerated $r-1$ ordered labeled partitions at each stage have to satisfy the criteria for compatible permutations as far as labels at various depths are concerned in order to correspond to a labeled (m, r) BTU.

2.33 Restricted Cycle Order Enumeration Problem

Given a set of $r-1$ unordered labeled partitions of m , to enumerate all possible distinct $r-1$ ordered labeled partitions, each of which refer a distinct choice of compatible permutation. $p_{i+1} \notin C(p_1, p_2, \dots, p_i)$ w.r.t. p_1, p_2, \dots, p_i for $1 \leq i \leq r-1$. Each of enumerated $r-1$ ordered labeled partitions at each stage have to satisfy the criteria for compatible permutations as far as labels at various depths are concerned in order to correspond to a labeled (m, r) BTU. For **Restricted Cycle Order Enumeration**, we impose the following additional constraints

1. Without loss of generality, we restrict $p_1 = I_m$.
2. $p_2 = \Psi(\beta_1)$. The first set of the $r-1$ ordered labeled partitions on m corresponding to choices for p_2 gets fixed due to this.
3. While choosing the second set of of the $r-1$ ordered labeled partitions of m corresponding to $p_3 \notin C(p_1, p_2)$ w.r.t. p_1, p_2 , we choose the first element

with the minimum label number from each ordered labeled partitions of m corresponding to $p_2 \notin C(p_1)$ w.r.t. p_1 .

We do not lose any non-isomorphic (m, r) BTU that could be constructed from the set of $r-1$ unordered labeled partitions of m in the enumeration process by imposing this constraint.

2.34 Micro-partitions Isomorphism Theorem for (m, r) BTUs

Enumeration of all non-isomorphic elements in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$

All non-isomorphic (m, r) BTUs in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ where $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ are enumerated at least once by

1. An exhaustive enumeration of Combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$.
2. An exhaustive enumeration of sets of $r-1$ Unordered labeled partitions for each choice of Combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$ enumerated in the previous step.
3. Restricted Cycle Order Enumeration of Sets of $r-1$ Ordered labeled partitions that result in compatible permutations corresponding to each Unordered labeled partition enumerated in the previous step.
4. We construct p_2, p_3, \dots, p_r corresponding to each set of $r-1$ Ordered labeled partitions enumerated in the previous step. ($p_1 = I_m$)

Proof Since any labeled (m, r) BTU with the first permutation $p_1 = I_m$ can be represented by a set of $r-1$ sets of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, mapping from micro-partitions to labels, and defined cycle orders for each partition component, it follows that for each non-isomorphic (m, r) BTU, there exists at least one set of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, one set of mapping from micro-partitions to labels, and one set of defined cycle orders for each partition component.

Without loss of generality, any non-isomorphic (m, r) BTU can be mapped to a

one set of micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$, to one set of mapping from micro-partitions to labels using **Restricted Cycle Order Enumeration**, and one set of defined cycle orders for each partition component.

2.35 Corollary Micro-partitions Isomorphism for $(m, 3)$ BTUs

All non-isomorphic $(m, 3)$ BTUs in $\Phi(\beta_1, \beta_2)$ can be enumerated by

1. An Exhaustive enumeration of Micro-partitions of β_2 w.r.t. β_1 .
2. An Exhaustive enumeration of 2 Unordered labeled partitions for each choice of Combinations of Micro-partitions of β_2 w.r.t. β_1 enumerated in the previous step.
3. Restricted Cycle Order Enumeration of Set of 2 Ordered labeled partitions that result in compatible permutations corresponding to each set of 2 Unordered labeled partitions enumerated in the previous step.
4. We construct p_2, p_3 corresponding to the set of 2 Ordered labeled partitions enumerated in the previous step. ($p_1 = I_m$).

2.36 Search Problem for BTU with best girth in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$

Given $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$, we exhaustively enumerate

1. Combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$.
2. A set of $r-1$ Unordered labeled partitions for each choice of Combinations of Micro-partitions of β_{i+1} w.r.t. β_i for $1 \leq i \leq r-1$ enumerated in the previous step.
3. Sets of $r-1$ Ordered labeled partitions that result in compatible permutations corresponding to each set of $r-1$ Unordered labeled partition enumerated in the previous step.
4. We construct p_2, p_3, \dots, p_r corresponding to the set of $r-1$ Ordered labeled partitions enumerated in the previous step. ($p_1 = I_m$).
5. We evaluate the girth for each of the enumerated/constructed (m, r) BTUs.
6. We choose the BTU with the best girth at the end of this process.

3 Direct Construction

3.1 Generalized Cycle Traversal for a permutation representation of a (m, r) BTU

If labels l_1 and l_2 occur at the same depth, labels l_2 and l_3 occur at the same depth, \dots , and finally labels l_x and l_1 occur at the same depth, in the permutation representation of a labeled (m, r) BTU, then there exists a cycle connecting the labels l_1, l_2, \dots, l_x .

3.2 Known Cycle Conjecture for a $(m, 2)$ BTU

The cycle lengths of a $(m, 2)$ BTU that is isomorphic to $\Psi(\beta)$ for some $\beta \in P_2(m)$ given by $\sum_{i=1}^y q_i = m$ are $\{2 * q_i\}; 1 \leq i \leq y$.

Proof A $(m, 2)$ BTU that is isomorphic to $\Psi(\beta)$ has no other cycles other than that of $\beta \in P_2(m)$ given by $\sum_{i=1}^y q_i = m$. The cycle length for a partition component q_i is $2 * q_i$. Hence, it follows that the cycle lengths are $\{2 * q_i\}; 1 \leq i \leq y$.

3.3 Maximum possible girth of a $(m, 2)$ BTU

The maximum possible girth of a $(m, 2)$ BTU is $2 * m$.

Proof This directly follows when we consider that every $(m, 2)$ BTU can be mapped to $\Psi(\beta)$ where $\beta \in P_2(m)$. It is clear that girth of a $(m, 2)$ BTU is $2 * \min(q_i); 1 \leq i \leq y$ where $\sum_{i=1}^y q_i = m$ represents $\beta \in P_2(m)$. Hence, it follows that the maximum possible girth of a $(m, 2)$ BTU is $2 * m$.

3.4 Upper Bounds Known partition component upper bound Conjecture

If $u = \min(q_{i,j}; 1 \leq j \leq y_i; 1 \leq i \leq r-1)$ where each partition β_i is given by $\sum_{j=1}^{y_i} q_{i,j} = m$ for $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ where each $u, q_{i,j} \in \mathbb{N}$, then the maximum possible girth of all (m, r) BTUs in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ is less than or equal to $2 * u$. This is an upper bound on the possible possible girth.

Proof This follows directly from the fact that there can be smaller cycles caused due to interactions between the partitions $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ and the maximum girth of all (m, r) BTUs in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ is less than or equal to $2 * u$, with strict equality when $r = 2$.

4 Micro-partition cycles

For a (m, r) BTU $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ where $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ given by $\{p_1, p_2, \dots, p_r\}; p_{i+1} \notin$

$C(p_1, p_2, \dots, p_i)$ for $1 \leq i \leq r-1$. If each β_i refers to $\sum_{j=1}^{y_i} q_{i,j} = m$ for $1 \leq i \leq r-1$, the micro-partition cycles are the cycles caused by interaction between the partition components of β_u and β_v where $u \neq v; 1 \leq u \leq r-1; 1 \leq v \leq r-1$ are the cycles caused due to interactions between the known cycles of β_u and β_v namely $\{q_{u,1}, q_{u,2}, \dots, q_{u,y_u}\}$ and $\{q_{v,1}, q_{v,2}, \dots, q_{v,y_v}\}$ and corresponding micro-partitions $\{x_{u,v,c,d}; 1 \leq c \leq y_u; 1 \leq d \leq y_v \text{ and } \{x_{v,u,d,c}; 1 \leq c \leq y_u; 1 \leq d \leq y_v\}$.

4.1 When do micro-partition cycles arise?

When $x_{u,v,c,d} \neq 1$ and $x_{u,v,c,d} \neq 0$ for some $\{c, d\}$ where $1 \leq c \leq y_u; 1 \leq d \leq y_v$, we have more than one point from partition component $q_{u,c}$ is used for creating partition component $q_{v,d}$, then we have an additional cycle referred to as micro-partition cycle.

4.2 Conjecture for Length of micro-partition cycle

For a (m, r) BTU in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ where $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$, the maximum possible length of micro-partition cycle is $\min\{2 * (q_{u,j}/x_{u,v,j,z} + t_u - 1)\}$; $x_{u,v,j,z} \geq 2; 1 \leq j < y_u; 1 \leq z \leq y_v$ where β_i refers to $\sum_{j=1}^{y_i} q_{i,j} = m$ for $1 \leq i \leq r-1; i \in \mathbb{N}$, where Generalized Micro-partition between $\beta_u, \beta_v \in P_2(m); u \neq v; 1 \leq u \leq r-1; 1 \leq v \leq r-1 : x_{u,v,j,z}$ where $1 \leq j \leq y_u; 1 \leq z \leq y_v$. t_u which is the number of partition components of β_u connected by the micro-partition cycle, $1 \leq t_u \leq y_u$.

Proof If $x_{u,v,c,d} \neq 1$ and $x_{u,v,c,d} \neq 0$ for some $\{c, d\}$ where $1 \leq c \leq y_u; 1 \leq d \leq y_v$, we have more than one point from partition component $q_{u,c}$ is used for creating partition component $q_{v,d}$, then we have a micro-partition cycle. Since the number of partition components of β_u connected by the micro-partition cycle is t_u , where $1 \leq t_u \leq y_u$, we have a smaller cycle which can take the maximum value $\{2 * (q_{u,j}/x_{u,v,c,d} + t_u - 1)\}$, by choosing the $x_{u,v,c,d}$ points appropriately. Hence, the maximum possible length of micro-partition cycle is $\min\{2 * (q_{u,j}/x_{u,v,j,z} + t_u - 1)\}$; $x_{u,v,j,z} \geq 2; 1 \leq j < y_u; 1 \leq z \leq y_v$.

4.3 Corollary for $(m, 3)$ BTU

For a $(m, 3)$ BTU in $\Phi(\beta_1, \beta_2)$ where $\beta_1, \beta_2 \in P_2(m)$, if micro-partition cycles exist, the minimum possible length of micro-partition cycle is $\min\{2 * (q_{1,j}/x_{1,j,z} + t_1 - 1)\}$ where the micro-partitions $x_{1,j,z} \geq 2; 1 \leq j < y_1; 1 \leq z \leq y_2$ where β_i refers to $\sum_{j=1}^{y_i} q_{i,j} = m$. t_1 which is the number of partition components of β_1 connected by the micro-partition cycle, $\max(t_1) = y_1$.

4.4 All cycles caused

Given a labeled (m, r) BTU with compatible permutations $\{p_1, p_2, \dots, p_r\}$ in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ where each $\beta_i \in P_2(m)$ refers to $\sum_{j=1}^{y_i} q_{i,j} = m$ for $1 \leq i \leq r-1$ we categorize its cycles in the following manner

1. Known cycles : These cycles refer to the cycles $\{q_{i,j}\}$ for $1 \leq j \leq y_i$ and $1 \leq i \leq r-1$ corresponding to $\beta_1, \beta_2, \dots, \beta_{r-1}$. We have $r * (r-1)/2$ partitions in total arising from all possible combinations of 2 permutations from the set of r permutations, out of which $r-1$ partitions are considered for known cycles.
2. Cycles due to other partitions: We consider generalized partitions $\alpha_{u,v} \in P_2(m)$ where $|(u-v)| \neq 0$, $|(u-v)| \neq 1$ and $1 \leq u \leq r; 1 \leq v \leq r$. The number of partitions considered here are $r * (r-1)/2 - (r-1) = r^2/2 - r/2 + 1$.
3. Micro-partition cycles if they arise due to interactions between the combinations of $r * (r-1)/2$ permutations.
4. Hidden cycles caused due to interaction of all the above cycles.

4.5 Strategy for girth maximization

In order to construct a (m, r) BTU with maximum girth, we choose

1. Partitions $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ such that the known cycles are maximized.
2. By maximizing the length of the micro-partition cycle, we obtain optimal parameters for $\beta_1, \beta_2, \dots, \beta_{r-1}$.
3. We search for permutations such that the cycles due to other partitions and hidden cycles caused due to interaction of all the above cycles is maximized.

4.6 Self Evident Fact About the Girth of a member of $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$

If $2 * v$ is the length of the minimum micro-partition cycle, and $u \in \mathbb{N}$ is the smallest partition component among given $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ i.e., $u = \min(q_{i,j}; 1 \leq j \leq y_i; 1 \leq i \leq r-1)$ where each β_i is given by $\sum_{j=1}^{y_i} q_{i,j} = m$, and $2 * w \in \mathbb{N}$ is the length of the smallest cycle caused due to interactions between the micro-partition cycles and cycles due to $\beta_1, \beta_2, \dots, \beta_{r-1}$, the girth of a member of $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ is $2 * \min(u, v, w)$ or $2 * \min(u, w)$ if there are no micro-partition cycles or $2 * u$ for $r = 2$, in which case the only cycles that arise due to one element of $P_2(m)$.

4.7 $(k, 2)$ BTU puncturing Conjecture

If a 0 element in a $(k, 2)$ BTU with cycle length $2 * k$ is changed to 1, then length of new minimum cycle within the sub-block $l \in \mathbb{N}$ satisfies $4 \leq l \leq k$ if k is an even positive integer and $4 \leq l \leq k + 1$ if k is an odd positive integer.

Proof Let us map the $(k, 2)$ BTU to a labeled directed graph with k vertices such that each vertex $i; 1 \leq i \leq k$ is connected to vertex $i + 1 \bmod k$. If the distance between two vertexes is defined as the length of shortest traversals in the same direction of the directed edges, it is clear that the maximum distance measured in terms of number of directed traversals from one vertex to the next, between two vertexes on this labeled directed graph is $k/2$ for even positive integers k and $(k + 1)/2$ for odd positive integers k , and the minimum distance measured in terms of number of directed traversals from one vertex to the next, between two vertexes on this labeled directed graph is 1. If a directed edge is connected between two vertexes of minimum distance of 1, this leads to a minimum cycle length of 4 on the matrix representation. If a directed edge is connected between two vertexes of maximum distance $k/2$ for even positive integers k and $(k + 1)/2$ for odd positive integers k , we get a cycle length of k for even positive integers k and $(k + 1)$ for odd positive integers k in the equivalent matrix representation.

Hence, the length of new minimum cycle within the sub-block $l \in \mathbb{N}$ satisfies $4 \leq l \leq k$ if k is an even positive integer and $4 \leq l \leq k + 1$ if k is an odd positive integer.

4.8 Three -one Conjecture

Let us consider a $(2 * k, 2)$ BTU constructed with $p_1 = I_{2 * k}$ and p_2 as per $\Psi((k, k))$, and if we have to additionally convert three 0 s in this BTU to 1 s, the girth is strictly less than $2 * k$.

Proof Girth of a $(2 * k, 2)$ BTU constructed with $p_1 = I_{2 * k}$ and p_2 as per $\Psi((k, k))$ is $2 * k$. Let us denote the $(2 * k, 2)$ BTU consisting of sub-matrices B_1 and B_2 each of which are $k \times k$ matrices that represent a constituent $(k, 2)$ BTU, and two $k \times k$ matrices referred to as $CB(1, 2)$ that shares its rows with B_1 and columns with B_2 and $CB(2, 1)$ that shares its rows with B_2 and columns with B_1 .

Let us consider different cases for placement of the three 1 s.

Case 1: If a 1 is placed inside either of the constituent $(k, 2)$ BTUs, by the previous theorem, the girth reduces to $k + 1$ if k is odd, and k if k is even.

Case 2: Three 1 s in $CB(1, 2)$

Let the positions of the three 1 s in $CB(1, 2)$ be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ such that $1 \leq x_i \leq k; 1 \leq y_i \leq k$ for $1 \leq i \leq 3$ and $x_1 \neq x_2; x_2 \neq x_3; x_3 \neq x_1; y_1 \neq$

$y_2; y_2 \neq y_3; y_3 \neq y_1$. Let us define $h(w_1, w_2, k) = \min\{|(w_1 - w_2)|, k - |(w_1 - w_2)|\}$

We can verify that traversals lengths between any two of the three points through B_1 are $2 * h(x_1, x_2, k) + 1$, $2 * h(x_2, x_3, k) + 1$ and $2 * h(x_3, x_1, k) + 1$.

We can verify that traversals lengths between any two of the three points through B_2 are $2 * h(y_1, y_2, k) + 1$, $2 * h(y_2, y_3, k) + 1$ and $2 * h(y_3, y_1, k) + 1$. The corresponding cycle lengths are $2 * h(x_1, x_2, k) + 2 * h(y_1, y_2, k) + 2$, $2 * h(x_2, x_3, k) + 2 * h(y_2, y_3, k) + 2$ and $2 * h(x_3, x_1, k) + 2 * h(y_3, y_1, k) + 2$.

If possible let the length of the minimum cycle be greater than or equal to $2 * k$, which implies that

$h(x_1, x_2, k) + h(y_1, y_2, k) \geq k - 1$, $h(x_2, x_3, k) + h(y_2, y_3, k) \geq k - 1$ and $h(x_3, x_1, k) + h(y_3, y_1, k) \geq k - 1$. This gives rise to a contradiction since the upper bound on maximum attainable value of

$\min(|(x_1 - x_2)|, |(x_2 - x_3)|, |(x_3 - x_1)|)$ is $k/3$ and similarly upper bound on maximum attainable value of

$\min(|(y_1 - y_2)|, |(y_2 - y_3)|, |(y_3 - y_1)|)$ is $k/3$.

Hence. The maximum attainable girth when three 1 s are placed in $CB(1, 2)$ is strictly less than $2 * k$.

Case 3: Three 1 s in $CB(2, 1)$

By repeating the argument for three 1 s in $CB(1, 2)$ we can show that the maximum attainable girth when three 1 s are placed in $CB(2, 1)$ is strictly less than $2 * k$.

Case 4: Two 1 s placed in $CB(1, 2)$ and one 1 placed in $CB(2, 1)$

Maximum value of traversal length from one point to another through B_1 is $k/3$. Maximum value of traversal length from one point to another through B_2 is $k/3$. Hence. The maximum attainable girth when two 1 s placed in $CB(1, 2)$ and one 1 placed in $CB(2, 1)$ is strictly less than $2 * k$.

Case 5: Two 1 s placed in $CB(2, 1)$ and one 1 placed in $CB(1, 2)$

By repeating the argument for two 1 s placed in $CB(1, 2)$ and one 1 placed in $CB(2, 1)$ we can show that the maximum attainable girth when two 1 s placed in $CB(2, 1)$ and one 1 placed in $CB(1, 2)$ is strictly less than $2 * k$.

Hence, given a $(2 * k, 2)$ BTU constructed with $p_1 = I_{2 * k}$ and p_2 as per $\Psi((k, k))$, and if we have to additionally convert three 0 s in this BTU to 1 s, the girth is strictly less than $2 * k$.

4.9 Upper bound on the maximum attainable girth for the case when no micro-partition cycles arises for a $(k^2, 3)$ BTU

No micro-partition cycles arise when $\beta_1, \beta_2 \in P_2(m)$ both correspond to the partition $\sum_{j=1}^k k = k^2$. Let us con-

struct a labeled $(k^2, 2)$ BTU with $p_1 = I_{k^2}$ and $p_2 \in S_{k^2}$ as per $\Psi(\beta_1)$. Let us consider the labeled $(k^2, 2)$ BTU as consisting of k constituent $(k, 2)$ BTUs which we refer to as sub-blocks numbers from $\{1, 2, \dots, k\}$ and $k^2 - k$ cross-blocks $CB(i, j)$ where $1 \leq i \leq k; 1 \leq j \leq k; i \neq j$. Each cross-block $CB(i, j)$ shares its rows with sub-block i and shares its columns with sub-block j .

Let $p_3 \in S_{k^2}$ be the permutation that maximizes the girth among all possible $(k^2, 3)$ BTUs in $\Psi(\beta_1, \beta_2)$.

There must exist at least one cross-block in each cross-block row that has two 1s from p_3 . There must exist at least one cross-block in each cross-block column that has two 1s from p_3 .

Hence, there must exist u, v such that $u \neq v$ and $1 \leq u \leq k; 1 \leq v \leq k$ such that $CB(u, v)$ and $CB(v, u)$ have two 1s and one 1 respectively.

If we consider a $2 * k \times 2 * k$ matrix consisting of a $(k, 2)$ BTU B_1 and $CB(u, v)$ in the same rows, and $(k, 2)$ BTU B_2 and $CB(v, u)$ in the same columns and consequently, B_1 and $CB(v, u)$ share the same columns and B_2 and $CB(u, v)$ share the same rows.

By using the previously proved result, girth is strictly less than $2 * k$.

Hence, the maximum attainable girth of $(k^2, 3)$ BTU with no micro-partition cycles is strictly less than $2 * k$.

4.10 Conjecture

The maximum attainable girth for a $(k^2, 3)$ BTU for the case when micro-partition cycles arise is greater than the maximum attainable girth for a $(k^2, 3)$ BTU for the case when no micro-partition cycles arise.

Proof For the case where no micro-partition cycles arise, let $p_3 \in S_{k^2}$ maximize the girth among all possible $(k^2, 3)$ BTUs in $\Psi(\beta, \beta)$ where $\beta \in P_2(m)$ corresponds to the partition $\sum_{j=1}^k k = k^2$ with $p_1 = I_{k^2}$ and $p_2 \in S_{k^2}$ as per $\Psi(\beta)$. Let us consider the labeled $(k^2, 2)$ BTU as consisting of k constituent $(k, 2)$ BTUs which we refer to as sub-blocks numbers from $\{1, 2, \dots, k\}$ and $k^2 - k$ cross-blocks $CB(i, j)$ where $1 \leq i \leq k; 1 \leq j \leq k; i \neq j$. Each cross-block $CB(i, j)$ shares its rows with sub-block i and shares its columns with sub-block j .

Let us choose the following $\beta_1, \beta_2 \in P_2(m)$ that correspond to $\sum_{j=1}^k k = k^2$ and $\sum_{j=1}^1 k^2 = k^2$ respectively. Without loss generality, p_3 is such that $CB(1, k-1), CB(2, 1), \dots, CB(k, k-2)$ and $CB(k-1, 1), CB(1, 2), \dots, CB(k-2, k)$ such that each pair of cross-blocks $\{CB(1, k-1), CB(k-1, 1)\}, \{CB(2, 1), CB(1, 2)\}, \dots,$

$\{CB(k, k-2), CB(k-2, k)\}$ have exactly two 1s between the two of them such that each 1 with coordinates $(x, y); 1 \leq x \leq k^2; 1 \leq y \leq k^2$

satisfies the constraint $|x-y| \geq k$.

We now replace p_2 with $q_2 \in S_{k^2}$ as per $\Psi(\beta_2)$.

Now, the cycle length due to $\{CB(1, k-1), CB(k-1, 1)\}$ and sub-blocks 1 and k is now $2 * k$ since the function for traversal length between two points (x_1, y_1) and (x_2, y_2)

is now $2 * |(x_1 - x_2)| + 1$ and $2 * |(y_1 - y_2)| + 1$ instead of $2 * \min\{|(x_1 - x_2)|, k - |(x_1 - x_2)|\} + 1$ and $2 * \min\{|(y_1 - y_2)|, k - |(y_1 - y_2)|\} + 1$.

The same is true for $\{CB(2, 1), CB(1, 2)\}, \dots, \{CB(k, k-2), CB(k-2, k)\}$.

Now, let us examine the situation that leads to the constraint on maximum attainable minimum cycle length when no micro-partitions cycles arise, and see that the maximum attainable minimum cycle length is better for when micro-partitions cycles arise.

Let us consider $CB(u, v)$ and $CB(v, u)$ with three 1s between both the cross-blocks in the pair.

Case 1: Three 1 in $CB(u, v)$ zero 1s in $CB(v, u)$.

Let the positions of the three 1s in $CB(u, v)$ be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ such that $1 \leq x_i \leq k; 1 \leq y_i \leq k$ for $1 \leq i \leq 3$ and $x_1 \neq x_2; x_2 \neq x_3; x_3 \neq x_1; y_1 \neq y_2; y_2 \neq y_3; y_3 \neq y_1$. Let us define $h_2(w_1, w_2) = |(w_1 - w_2)|$

We can verify that traversals lengths between any two of the three points through B_1 are $2 * h_2(x_1, x_2) + 1$, $2 * h_2(x_2, x_3) + 1$ and $2 * h_2(x_3, x_1) + 1$.

We can verify that traversals lengths between any two of the three points through B_2 are $2 * h_2(y_1, y_2) + 1$, $2 * h_2(y_2, y_3) + 1$ and $2 * h_2(y_3, y_1) + 1$. The corresponding cycle lengths are $2 * h_1(x_1, x_2) + 2 * h_2(y_1, y_2) + 2$, $2 * h_2(x_2, x_3) + 2 * h_2(y_2, y_3) + 2$ and $2 * h_2(x_3, x_1) + 2 * h_2(y_3, y_1) + 2$.

Thus, the length of the minimum cycle is increased compared to the previous traversal length $2 * h(y_3, y_1, k) + 1$ where $h(y_3, y_1, k) = \min\{|(y_1 - y_2)|, k - |(y_1 - y_2)|\}$.

Case 2: We can similarly show that the one Cross-block has two 1s and other in the pair has one 1s, the length of the minimum cycle is increased for the case when micro-partition cycles arise.

Similarly, we can also show that for the case when $CB(u, v)$ and $CB(v, u)$ with two 1s between both the cross-blocks in the pair, has the length of the maximum cycle increased.

Given any points corresponding to p_3 , we can show that the traversal length increases for the considered case when micro-partition cycles arise.

Hence, the maximum attainable girth for a $(k^2, 3)$ BTU for the case when micro-partition cycles arise is greater than the maximum attainable girth for a $(k^2, 3)$ BTU for the case when no micro-partition cycles arise.

4.11 Girth maximum Conjecture for $(k^2, 3)$ BTU

If $k \in \mathbb{N}; k > 3$ there exists a $(k^2, 3)$ BTU in $\Phi(\beta_1, \beta_2)$ with maximum girth among all $(k^2, 3)$ BTUs where each

$\beta_i \in P_2(k^2)$ refers to $\sum_{j=1}^{k^{3-1-i}} \{k^i\} = k^2$ for $1 \leq i \leq 2$. **Proof** Let $\beta_1, \beta_2 \in P_2(b * k^2)$ be of the form $\sum_{j=1}^{y_i} q_{i,j} = k^2$ for $1 \leq i \leq 2$. Since we have proved that the case where no micro-partition cycles arise produces lesser minimum cycle length than the case where micro-partition cycles arise, let us maximize the length of micro-partition cycles.

Micro-partition cycles and cycles due to β_1, β_2 can produce other interacting cycles that are of smaller length. Hence, let us maximize the length of the minimum micro-partition cycle. Since the length of the minimum micro-partition cycle would be less than or equal to $\min\{2 * (q_{1,j}/x_{1,j,z} + t_1 - 1)\}$, for the case for micro-partition cycles maximized, we obtain $q_{2,1} = k^2; y_1 = k; q_{1,j} = k; y_2 = 1$ and micro-partitions $x_{1,2,j,1} = k$ for $1 \leq j \leq k$, $x_{2,1,1,j} = k$ for $1 \leq j \leq k$

Micro-partition cycles are

$$2 * (k^2/k + 1 - 1) = 2 * k \text{ and}$$

$$2 * (k/k + k - 1) = 2 * k.$$

Starting with $p_1 = I_{k^2}$ and $p_2 \in S_{k^2}$ as per $\Psi(\beta_1)$, we can choose $p_3 \in S_{k^2}$ by considering the interactions between the micro-partition cycles and known cycles, i.e., one cycle of length $2 * k^2$ and k cycles of length $2 * k$, we construct a girth maximum $(k^2, 3)$ BTU. Given $k \in \mathbb{N}$, thus \exists a BTU with maximum girth among all $(k^2, 3)$ BTUs in $\Phi(\beta_1, \beta_2)$ where $\beta_1, \beta_2 \in P_2(m)$ correspond to $\sum_{j=1}^k k = k^2$ and $\sum_{j=1}^1 k^2 = k^2$ respectively.

4.12 Conjecture

If $k \in \mathbb{N}; k > 3$ and $b \in \mathbb{N}; b^2 < k$ there exists a $(b * k^2, 3)$ BTU in $\Phi(\beta_1, \beta_2)$ with maximum girth among all $(b * k^2, 3)$ BTUs where each $\beta_i \in P_2(b * k^2)$ refers to $\sum_{j=1}^{k^{3-1-i}} \{b * k^i\} = b * k^2$ for $1 \leq i \leq 2$.

4.13 Notation for scaling of a partition

Scaling of a partition $\alpha \in P_2(m)$ which refers to $\sum_{j=1}^y q_j = m$ by k is denoted by $k * \alpha \in P_2(k * m)$ which refers to the partition $\sum_{j=1}^{k*y} q_j = k * m$.

4.14 Conjecture for girth maximum $(b * k^{r-1}, r)$ BTU

If $k, r \in \mathbb{N}; k > r$ and $b = \prod_{i=1}^{r-1} b_i$ such that $b_i \in \mathbb{N}$ for $1 \leq i \leq r-1$ satisfy $b_1 \leq b_2 \leq \dots \leq b_{r-1} < k$ and $b_1 = 1$, there exists a $(b * k^{r-1}, r)$ BTU in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ with maximum girth among all $(b * k^{r-1}, r)$ BTUs where each $\beta_i \in P_2(b * k^{r-1})$ refers to $\sum_{j=1}^{k^{r-1-i}} \{b * k^i\} = b * k^{r-1}$ for $1 \leq i \leq r-1$.

Proof We prove the above statement using the principle of mathematical induction. For $r = 3$, with $p_1 = I_{b*k^2}$ and p_2 as per $\Psi(\beta_1)$ where $\beta_1 \in P_2(b * k^2)$ refers to

$\sum_{j=1}^k \{b * k^1\} = b * k^2$, and $\beta_2 \in P_2(b * k^2)$ refers to $\sum_{j=1}^1 \{b * k^2\} = b * k^2$.

Since $(b * k^1, 2)$ BTU with $\sum_{j=1}^1 \{b * k^1\} = b * k^1$ has maximum girth among all $(b * k^1, 2)$ BTUs, we choose $p_3 \in S_{b*k^2}$ so that partition between p_2 and p_3 is $\sum_{j=1}^1 \{b * k^2\} = b * k^2$, such that we get maximum girth, clearly there exists a BTU with maximum girth in $\Phi(\beta_1, \beta_2)$. Hence the statement is proven for $r = 3$.

Let us assume that the statement is true for $r = l$, we need to prove that it is also true for $r = l + 1$.

We assume that there exists $(b * k^{l-1}, l)$ BTU with maximum girth in $\Phi(\lambda_1, \lambda_2, \dots, \lambda_{l-1})$ where each $\lambda_i \in P_2(b * k^{l-1})$ refers to $\sum_{j=1}^{k^{l-1-i}} \{b * k^i\} = b * k^{l-1}$ for $1 \leq i \leq l-1$.

We need to prove that there exists $(b * k^l, l + 1)$ BTU with maximum girth in $\Phi(\beta_1, \beta_2, \dots, \beta_l)$ where each $\beta_i \in P_2(b * k^l)$ refers to $\sum_{j=1}^{k^{l-i}} \{b * k^i\} = b * k^l$ for $1 \leq i \leq l$.

By scaling each $\lambda_i \in P_2(b * k^{l-1})$ in the set $\{(\lambda_1, \lambda_2, \dots, \lambda_{l-1})\}$ by a factor of k to $\{(k * \lambda_1, k * \lambda_2, \dots, k * \lambda_{l-1})\}$, we get $\sum_{j=1}^{k^{l-i}} \{b * k^i\} = b * k^l$ for $1 \leq i \leq l$ which are nothing but $\{(\beta_1, \beta_2, \dots, \beta_{l-1})\}$.

We use k instances of the girth maximum $(b * k^{l-1}, l)$ BTU, and by choosing $p_{l+1} \in S_{b*k^l}$ so that partition between p_l and p_{l+1} is $\sum_{j=1}^1 \{b * k^l\} = b * k^l$ such that we get maximum girth by maximizing length of the micro-partition cycles,

$q_{l-1,1} = b * k^l; y_{l-1} = k; q_{l,1} = b * k^{l+1}; y_l = 1$ and generalized Micro-partitions $x_{l,l-1,j,1} = k$ for $1 \leq j \leq k$ and $x_{l-1,l,1,j} = k$ for $1 \leq j \leq k$.

Hence, the micro-partition cycles are $2 * (k^2/k + 1 - 1) = 2 * k$ and $2 * (k/k + k - 1) = 2 * k$. We can show that the case where no micro-partition cycles arise leads to cycle length strictly less than $2 * k$, and hence maximizing length of the micro-partition cycles leads to maximum girth.

We hence obtain the $(b * k^l, l + 1)$ BTU with maximum girth which clearly lies in $\Phi(\beta_1, \beta_2, \dots, \beta_l)$. Hence the statement is true for $r = l + 1$ and hence by the principle of finite induction, the statement is true for all $r \geq 3; r \in \mathbb{N}$.

4.15 Algorithm for girth maximizing (m, r) BTU

Assumptions: We assume that $m \in \mathbb{N}$ is a composite number.

1. We factorize $m = k^{r-1} * b$ where $k, b \in \mathbb{N}$ such that b is minimized.
2. Choose $\alpha_i \in P_2(k^i)$ as $\sum_{j=1}^1 \{k^i\} = k^i$ for $1 \leq i \leq r-1$.

α_1	$\sum_{j=1}^1 \{k\} = k$
α_2	$\sum_{j=1}^1 \{k^2\} = k^2$
\dots	\dots
α_{r-1}	$\sum_{j=1}^1 \{k^{r-1}\} = k^{r-1}$

β_1	$\sum_{j=1}^{k^{r-2}} \{b * k\} = b * k^{r-1} = m$
β_2	$\sum_{j=1}^{k^{r-3}} \{b * k^2\} = b * k^{r-1} = m$
\dots	\dots
β_{r-1}	$\sum_{j=1}^1 \{b * k^{r-1}\} = b * k^{r-1} = m$

The corresponding $\gamma_i \in P_2(k^{r-1})$ are chosen as $\sum_{j=1}^{k^{r-1-i}} \{k^i\} = k^{r-1}$ for $1 \leq i \leq r-1$.

γ_1	$\sum_{j=1}^{k^{r-2}} \{k\} = k^{r-1}$
γ_2	$\sum_{j=1}^{k^{r-3}} \{k^2\} = k^{r-1}$
\dots	\dots
γ_{r-1}	$\sum_{j=1}^1 \{k^{r-1}\} = k^{r-1}$

The corresponding $\beta_i \in P_2(m)$ are chosen as $\sum_{j=1}^{k^{r-1-i}} \{b * k^i\} = b * k^{r-1} = m$ for $1 \leq i \leq r-1$.

After computation of the optimal partitions, the search for a girth maximum BTU involves the following steps.

1. We construct a girth maximum $(b * k, 2)$ BTU.
2. We search for a girth maximum $(b * k^2, 3)$ BTU.
3. We search for a girth maximum $(b * k^3, 4)$ BTU.
4. We search for a girth maximum $(b * k^{r-1}, r)$ BTU.

4.16 Conjecture

In order to construct a (m, r) BTU with best girth where $k^{r-1} = m/b$; $k = (m/b)^{1/r-1}$; $k \in \mathbb{N}$ where $m = k^{r-1} * b$ where $k, b \in \mathbb{N}$ such that b is minimized, β_i is chosen as $\sum_{j=1}^{k^{r-1-i}} \{b * k^i\} = b * k^{r-1} = m$ for $1 \leq i \leq r-1$, it is sufficient to construct $(k^2, 3)$ BTU with best girth, and use this as a template for making the rest of the connections as described by the following hierarchy of girth maximum BTUs

i	β_i	BTU
1	$\sum_{j=1}^{k^{r-2}} b * k^1 = m$	$(m, 2)$
2	$\sum_{j=1}^{k^{r-3}} \{b * k^2\} = b * k^{r-1} = m$	$(m, 3)$

$r - 2$	$\sum_{j=1}^k \{b * k^{r-2}\} = b * k^{r-1} = m$	$(m, r - 1)$
$r - 1$	$\sum_{j=1}^1 \{b * k^{r-1}\} = b * k^{r-1} = m$	(m, r)

5 Search Problem for finding $(b * k^2, 3)$ BTU with best girth where $k \in \mathbb{N}$

For $r = 3$, with $p_1 = I_{b * k^2}$ and $p_2 \in S_{b * k^2}$ as per $\Psi(\beta_1)$ where $\beta_1 \in P_2(b * k^2)$ refers to $\sum_{j=1}^k \{b * k^1\} = b * k^2$, and $\beta_2 \in P_2(b * k^2)$ refers to $\sum_{j=1}^1 \{b * k^2\} = b * k^2$, to find $p_3 \in S_{b * k^2}$ such that the labeled BTU $\{p_1, p_2, p_3\}$ has maximum girth among all $(b * k^2, 3)$ BTUs.

5.1 Algorithm to generate optimal partitions for a given value of k and r

The following algorithm generate optimal partitions $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(k^{r-1})$ for a given value of k and

r such that the girth maximum (k^{r-1}, r) BTU lies in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$. $m = k$;
for($i = 1; i \leq r - 1; i++$) {
 β_i refers to $\sum_{j=1}^1 m$;
for($z = 1; z < i; z++$) {
 $k * \beta_z$; //scale partition β_z by k
}
 $m = k * m$;
}

6 Search Problem for finding $(b * k^{r-1}, r)$ BTU with best girth where $k \in \mathbb{N}$

$p_1 = I_{b * k^2}$ and $p_2 \in S_{b * k^2}$ as per $\Psi(\beta_1)$ where $\beta_1 \in P_2(b * k^2)$ refers to $\sum_{j=1}^{k^{r-2}} b * k^1 = b * k^{r-1}$, $\beta_2 \in P_2(b * k^2)$ refers to $\sum_{j=1}^{k^{r-3}} \{b * k^2\} = b * k^{r-1}$, ... and $\beta_{r-1} \in P_2(b * k^2)$ refers to $\sum_{j=1}^1 \{b * k^{r-1}\} = b * k^{r-1}$, we need to find $p_3, \dots, p_{r-1} \in S_{b * k^{r-1}}$ such that the labeled BTU $\{p_1, p_2, p_3, \dots, p_{r-1}\}$ has maximum girth among all $(b * k^{r-1}, r)$ BTUs.

7 Open Questions on girth maximum (m, r) BTU

1. What is the maximum attainable girth for a (m, r) BTU?
2. How do we construct an optimal search problem for finding a girth maximum (m, r) BTU ?
3. What is the computational complexity of the search problem for finding a girth maximum (m, r) BTU ?

8 CONCLUSION

This paper describes the optimal partition parameters for a girth maximum (m, r) BTU. We mathematically prove results for optimal parameters $\beta_1, \beta_2, \dots, \beta_{r-1} \in P_2(m)$ such that the girth maximum (m, r) BTU lies in $\Phi(\beta_1, \beta_2, \dots, \beta_{r-1})$ and create a framework for specifying a search problem for finding the girth maximum (m, r) BTU. We also raise some open questions on girth maximum (m, r) BTU.

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